

# Parallel versus Conventional Elastoplasticity

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*Conventional elastoplastic laws are too poor to model correctly the phenomena of elastic-plastic transition and of failure. Moreover, the evolution of elastic and plastic anisotropy is hard to describe and harder to verify experimentally. On the other hand, the simulation of whole aggregates of crystals is extremely time-consuming. In this paper an intermediate way is proposed, namely the construction of an elastoplastic material element by parallel connexion of few constituents which reflect essential aspects of the behavior of b.c.c. crystals. The description makes use of tensors which are invariant under the superposition of a rigid body rotation. An implicit integration scheme is presented and the formation of elastic and plastic anisotropy is demonstrated.*

## 1 The Aim of the Investigation

An elastic-plastic constitutive law which is to meet the requirements of advanced simulation (e.g. of deep drawing of steel or aluminum sheets) should be able to reproduce the following phenomena:

1. Elastic behavior
2. Plastic behavior
3. Hardening
4. Bauschinger effect
5. Elastic and plastic anisotropy
6. Change of anisotropy during the deformation process
7. Elastic-plastic transition zone
8. Limit to formability due to localization

Classical constitutive laws of the Prandtl-Reuss type are only able to cover the first three items. Extensions on a purely phenomenological basis are not satisfactory. They require ad hoc assumptions on internal variables and their evolution, the shape of one or several yield surfaces and their change during plastic deformation. But these assumptions can hardly be verified by experiments over a sufficiently large range of deformation. So this approach has come to a deadlock and does no longer represent the state of the art.

More recent approaches start from the idea that the complex macroscopic behavior of a polycrystalline material is the result of the co-operation of many single crystals on a mesoscale. Thus only the relatively simple constitutive behavior of single crystals has to be modelled and the bulk phenomena are the outcome of an analysis of an aggregate of such crystals. This can be done by the application of homogenization procedures to a representative volume element (RVE) containing a large number of crystals. A complete FEM analysis of the RVE yields reliable results but simplifying procedures (assumptions on the deformation or the stress field within the RVE or a self-consistent approach) are also applied.

But, after all, during the simulation of a forming process, the material behavior must be traced at each point of integration. Which is then the substitute for the classical constitutive law?

One idea is to use information from a RVE analysis in order to obtain reliable constitutive laws of the classical type (with yield surfaces, flow rules etc.) But the behavior of an aggregate is too complex to fit into such a corset, so that this approach has not been successful.

Another idea is to introduce the single crystals directly at the point of integration. This can only be done within acceptable computation times, if severe simplifications (e.g. constant straining of all crystals — equivalent to a parallel connexion) are introduced. In this way, the use of a large number of crystals (e.g. 256) was tested by Beaudoin et al. (1995). Investigations of this kind have also been carried out in a viscoplastic context by Nakamachi and Dong (1996).

One should be aware that the representation of crystal behavior at each point of integration will necessarily be coarse since the spatial arrangement and the interaction of the crystals as well as the shape of the grains and the influence of the grain boundaries can hardly be taken into account. So it seems reasonable to examine whether the essentials of a crystalline aggregate can perhaps be modelled by the parallel connexion of a small number of constituents. This is the goal of the following investigation. Furthermore, we will see how such a description fits into the framework of the theory of simple materials.

Our presentation will be based on the following assumptions:

- A purely mechanical theory is adopted
- The material behavior is rate-independent
- There exists an elastic range for each single crystal
- The elastic deformations of the single crystal are small and the elastic behavior is linear
- Any hardening of the single crystals is intentionally excluded in order to elucidate the essential distinctions of the parallel connexion and the classical approach

## 2 The Constitutive Behavior of a Single Crystal

We base our description of large deformations of a crystal on the multiplicative decomposition

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p \quad (1)$$

of the local transplacement  $\mathbf{F}$  (also called deformation gradient) into an elastic and a plastic part. (This sometimes suspected approach can rigourously be founded on the concepts of a tangent space of the body and a lattice space of the crystal without introducing any reference placement in an observer frame – cf. Krawietz (1986). We shall not make use of these concepts, however, since they have not yet been received by the engineering community.) We denote the actual density by  $\varrho$ , the Cauchy stress by  $\mathbf{T}$  and call  $\mathbf{T}/\varrho$  the modified Cauchy stress. The right Cauchy-Green tensor  $\mathbf{C}$  und Green's strain tensor  $\mathbf{E}$  are defined by

$$\mathbf{C} = \mathbf{1} + 2\mathbf{E} = \mathbf{F}^T \mathbf{F} \quad (2)$$

Our constitutive equations will exclusively be formulated with tensors which are invariant under a superimposed rigid body rotation. This is the case with  $\mathbf{F}_p$ ,  $\mathbf{C}$  and  $\mathbf{E}$  as well as with the modified Kirchhoff stress  $\mathbf{Z}$  which results from the pull-back of the modified Cauchy stress to the reference placement of the material:

$$\mathbf{Z} = \mathbf{F}^{-1} \frac{\mathbf{T}}{\varrho} \mathbf{F}^{-T} \quad (3)$$

The elastic right Cauchy-Green tensor  $\mathbf{C}_e$  and Green's elastic strain tensor  $\mathbf{E}_e$  are defined by

$$\mathbf{C}_e = \mathbf{1} + 2\mathbf{E}_e = \mathbf{F}_e^T \mathbf{F}_e = \mathbf{F}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} \quad (4)$$

The pull-back of the modified Cauchy stress to the reference placement of the lattice shall be called the modified lattice stress

$$\mathbf{Z}_e = \mathbf{F}_e^{-1} \frac{\mathbf{T}}{\varrho} \mathbf{F}_e^{-T} = \mathbf{F}_p \mathbf{Z} \mathbf{F}_p^T \quad (5)$$

The time derivative of equation (4) yields the kinematical relation

$$2\dot{\mathbf{E}}_e = \mathbf{F}_p^{-T} \dot{\mathbf{C}} \mathbf{F}_p^{-1} - \mathbf{F}_p^{-T} \dot{\mathbf{F}}_p^T \mathbf{C}_e - \mathbf{C}_e \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \quad (6)$$

The linear elastic behavior of the lattice is characterized by

$$\mathbf{Z}_e = \mathcal{C} : \mathbf{E}_e \quad (7)$$

with a symmetric fourth order tensor  $\mathcal{C}$  which ensures the existence of an elastic strain energy

$$W = \frac{1}{2} \mathbf{E}_e : \mathcal{C} : \mathbf{E}_e \quad (8)$$

Combining equation (7) with equation (4) we find

$$\mathbf{Z}_e = \mathcal{C} : \frac{1}{2}(\mathbf{F}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} - \mathbf{1}) \quad (9)$$

The plastic deformation of the crystal occurs by sliding on discrete planes in the form

$$\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \sum_j \kappa_j \mathbf{m}_j \otimes \mathbf{n}_j \quad (10)$$

where  $\kappa_j$  denotes the velocity of shearing and  $\mathbf{n}_j$  and  $\mathbf{m}_j$  are unit vectors normal to the plane of sliding and in the direction of sliding, respectively, both measured in the (stress free) reference placement of the lattice.

In the light of Noll's (1972) theory of simple materials the actual state of the crystal is given by the external variable of state  $\mathbf{C}$  and the internal variable of state  $\mathbf{F}_p$  the evolution of which is governed by equation (10). Large deformations of crystals are often treated by means of orthogonal tensors (determined by Eulerian angles, quaternions etc.) which give the rotation of the lattice with respect to the observer. In reality, the lattice is not simply rotating but undergoes elastic deformations together with changes of its orientation. While rotations are indispensable in rigid body mechanics, they are of no use whatsoever in the mechanics of deformable media and we shall not employ them. The lattice is accessible to us in its reference placement where the crystal axes are constant, the tensor  $\mathbf{F}_p^{-1}$  gives us the reference placement of the matter which actually forms the lattice and the tensor  $\mathbf{F}_e = \mathbf{F} \mathbf{F}_p^{-1}$  may inform us how the elastically strained lattice actually presents itself to the observer.

The work per unit mass and unit time is given by — note equations (5), (6), (7), (8) —

$$\dot{j} = \mathbf{Z} : \dot{\mathbf{E}} = \mathbf{Z}_e : \dot{\mathbf{E}}_e + \mathbf{C}_e \mathbf{Z}_e : \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \dot{W} + \delta \quad (11)$$

with the power of dissipation — note equation (10) —

$$\delta = \mathbf{C}_e \mathbf{Z}_e : \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \sum_j \kappa_j \bar{\tau}_j \quad (12)$$

where the abbreviation  $\bar{\tau}_j$  — called modified shear stress (which in case of small elastic strains is nearly the Schmid stress divided by the density) — stands for

$$\bar{\tau}_j = \mathbf{m}_j \mathbf{C}_e \mathbf{Z}_e \mathbf{n}_j \quad (13)$$

States are only admissible if they lie within the elastic range characterized by the restrictions

$$f_j = \bar{\tau}_j - y \leq 0 \quad (14)$$

for each sliding mechanism, where the critical value  $y$  is assumed to be the same for all mechanisms and not subject to any change by hardening. A sliding of the mechanism  $j$  shall only be possible, if the corresponding modified shear stress satisfies the yield condition

$$f_j = \bar{\tau}_j - y = 0 \quad (15)$$

While the dissipation inequality merely requires  $\delta \geq 0$ , we follow von Mises (1928) postulating that the power of dissipation be maximal:

$$\delta \geq \delta^* \quad \implies \quad (\mathbf{C}_e \mathbf{Z}_e - \mathbf{C}_e^* \mathbf{Z}_e^*) : \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \geq 0 \quad (16)$$

Plastic yield actually takes place at the unstarred state on the yield surface while the starred state is any state within the elastic range. Inserting equation (12) gives

$$\delta - \delta^* = \sum_j \kappa_j (\bar{\tau}_j - \bar{\tau}_j^*) = \sum_j \kappa_j (y - \bar{\tau}_j^*) \geq 0 \quad (17)$$

We adopt the postulate

$$\kappa_j \geq 0 \quad (18)$$

which because of equation (14) is sufficient for the validity of equation (17).

Restricting our attention to cubic crystals we represent the elastic stiffness of the lattice by

$$\varrho_0 \mathcal{C} = 2G \mathcal{I} + \frac{\nu E}{(1+\nu)(1-2\nu)} \mathbf{1} \otimes \mathbf{1} + \left( \frac{E}{1+\nu} - 2G \right) \sum_{i=1}^3 \mathbf{g}_i \otimes \mathbf{g}_i \otimes \mathbf{g}_i \otimes \mathbf{g}_i \quad (19)$$

with  $\varrho_0$ ,  $E$ ,  $G$ ,  $\nu$  denoting the density of the unstressed state, the moduli of tension and shear and Poisson's ratio, respectively.  $\mathcal{I}$  is the identical mapping on the space of symmetric tensors and  $\mathbf{g}_i$  are the orthonormal vectors of the crystal axes in the reference placement of the lattice.

The sliding directions  $\mathbf{m}_j$  of a body-centered cubic crystal are the four space diagonals which include equal angles with the three crystal axes. Every plane which contains a space diagonal may be a plane of sliding (pencil glide). We introduce the modified shear vector  $\mathbf{s}_j$  on a plane which is perpendicular to the sliding direction  $\mathbf{m}_j$  by

$$\mathbf{s}_j = \mathbf{m}_j \mathbf{C}_e \mathbf{Z}_e (\mathbf{1} - \mathbf{m}_j \otimes \mathbf{m}_j) \approx \mathbf{m}_j \mathbf{Z}_e (\mathbf{1} - \mathbf{m}_j \otimes \mathbf{m}_j) = \mathbf{m}_j \mathbf{Z}'_e (\mathbf{1} - \mathbf{m}_j \otimes \mathbf{m}_j) \quad (20)$$

Since the elastic lattice deformation is small, we made use of the approximation  $\mathbf{C}_e = \mathbf{1} + 2\mathbf{E}_e \approx \mathbf{1}$  and noticed that then  $\mathbf{s}_j$  does not depend on the isotropic part of  $\mathbf{Z}_e$  but only on the deviatoric part  $\mathbf{Z}'_e$ . The condition (14) becomes — with equation (13) and because of  $\mathbf{m}_j \cdot \mathbf{n}_j = 0$  —

$$f_j = \mathbf{s}_j \cdot \mathbf{n}_j - y \leq 0 \quad (21)$$

The scalar product  $\mathbf{s}_j \cdot \mathbf{n}_j$  attains its maximum value  $|\mathbf{s}_j|$  for the choice

$$\mathbf{n}_j = \frac{\mathbf{s}_j}{|\mathbf{s}_j|} = \frac{\mathbf{m}_j \mathbf{Z}'_e (\mathbf{1} - \mathbf{m}_j \otimes \mathbf{m}_j)}{|\mathbf{m}_j \mathbf{Z}'_e (\mathbf{1} - \mathbf{m}_j \otimes \mathbf{m}_j)|} \quad (22)$$

Therefore sliding will be possible on the plane perpendicular to this  $\mathbf{n}_j$  if the condition

$$f_j = |\mathbf{s}_j| - y = 0 \quad (23)$$

is valid, while the strict inequality still holds in equation (21) for any other plane.

An explicit integration of the evolution equation (10) — based on a condition of consistency — would require an extremely small time step for reasons of stability. Therefore, we construct an implicit integration algorithm by assuming that during the finite time increment  $(t_0, t_0 + \Delta t)$

- the direction of the deviator  $\mathbf{Z}'_e$  of the modified lattice stress and hence the four unit normal vectors  $\mathbf{n}_j$  according to equation (22) and also
- the ratio of the four shearing velocities  $\kappa_j$

remain constant. Thus we set

$$\kappa_j = \dot{k}(t) \Delta \gamma_j \quad j = 1 \dots 4 \quad \text{with} \quad k(t_0) = 0, \quad k(t_0 + \Delta t) = 1 \quad (24)$$

and hence

$$\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \sum_{j=1}^4 \kappa_j \mathbf{m}_j \otimes \mathbf{n}_j = \dot{k}(t) \sum_{j=1}^4 \Delta \gamma_j \mathbf{m}_j \otimes \mathbf{n}_j \stackrel{\text{def}}{=} -\dot{k}(t) \mathbf{K} \quad (25)$$

According to our assumptions the tensor  $\mathbf{K}$  is constant during the increment so that the solution of this differential equation can be written down:

$$\mathbf{F}_p(t) = e^{-k(t) \mathbf{K}} \mathbf{F}_p(t_0) \quad \implies \quad \mathbf{F}_p(t_0 + \Delta t) = e^{-\mathbf{K}} \mathbf{F}_p(t_0) \quad (26)$$

Since  $\mathbf{K}$  is a deviator, this exponential mapping ensures an isochoric plastic deformation due to the identity  $\det \exp \equiv \exp \operatorname{tr}$ .

In the sense of the backward Eulerian integration scheme we evaluate the modified lattice stress at the end of the increment — see equation (9) —

$$\mathbf{Z}_e(t_0 + \Delta t) = \mathcal{C} : \frac{1}{2} \left( e^{\mathbf{K}^T} \mathbf{F}_p(t_0)^{-T} \mathbf{C}(t_0 + \Delta t) \mathbf{F}_p(t_0)^{-1} e^{\mathbf{K}} - \mathbf{1} \right) \quad (27)$$

Thus  $\mathbf{Z}_e$  and hence — according to equation (22) — the four normal vectors  $\mathbf{n}_j$  depend on  $\mathbf{K}$  and a combination with equation (25) gives

$$\mathbf{K} = - \sum_{j=1}^4 \Delta\gamma_j \mathbf{m}_j \otimes \mathbf{n}_j(\mathbf{K}) \quad (28)$$

This is a nonlinear deviator equation — equivalent to eight scalar equations — for the determination of the eight components of the deviator  $\mathbf{K}$  and of the four incremental angles of shear  $\Delta\gamma_j$ . Another four scalar equations are obtained as follows. If a sliding mechanism is active, then the yield condition (23)

$$|\mathbf{s}_j(\mathbf{K})| = y \quad (29)$$

must be satisfied; if it is inactive then

$$\Delta\gamma_j = 0 \quad (30)$$

must hold. The selection of active and inactive mechanisms is done correctly if the inequality

$$\Delta\gamma_j \geq 0 \quad (31)$$

according to equations (18), (24) is valid for any active mechanism and the inequality

$$|\mathbf{s}_j(\mathbf{K})| \leq y \quad (32)$$

for any inactive one. So 16 possibilities have to be tested in the worst case.

Figure 1 shows the typical behavior of such a crystal under a constant stretching from the unstressed state. (The lattice constants are taken from  $\alpha$ -Fe.) Initially there is pure elastic straining and finally steady state plastic flow. But in between we notice a transition phenomenon. Not all of the four sliding mechanisms are activated at the onset of plastic yield. This leads to a rearrangement of stress and elastic lattice deformation till further mechanisms become active. The final stress is situated at a vertex of the boundary of the elastic range. This behavior of individual crystals gives rise to a gradual elastic-plastic

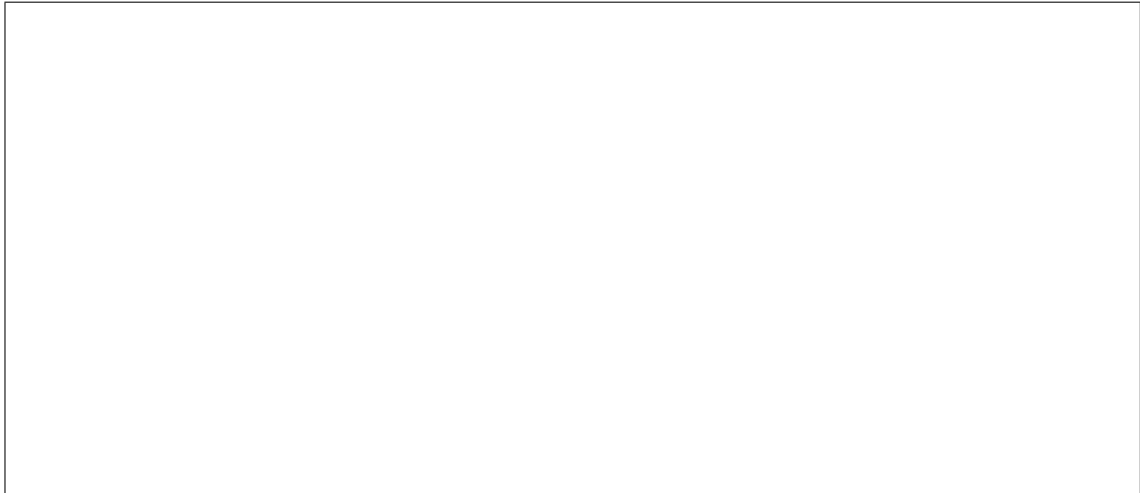


Figure 1. Single crystal: Elastic-plastic transition under stretching  $\mathbf{D} = \operatorname{diag}(2, 1, -3)$ ; first and second principal axes inclined under  $45^\circ$  with respect to the crystal axes.

transition of polycrystals. Moreover the polycrystal behavior is rather soft since several sliding mechanisms are active in each crystal. These features are not modelled by classical constitutive laws.

We formulate such a law of the Prandtl-Reuss type for later use. The equations (1) to (9) remain valid with an isotropic stiffness  $\mathcal{C}$  in equation (7), the yield condition (23) is replaced by

$$f = |\mathbf{Z}'_e| - y = 0 \quad (33)$$

and, instead of equation (10), we have in the case of yielding

$$\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \kappa \mathbf{Z}'_e \quad \text{with} \quad \kappa = \frac{1}{2y^2} \mathbf{Z}'_e : \mathbf{F}_p^{-T} \dot{\mathbf{C}} \mathbf{F}_p^{-1} \quad (34)$$

where  $\kappa$  has been inferred from the condition of consistency. Since the direction of plastic flow is dictated by  $\mathbf{Z}'_e$ , the behavior is rather stiff, and hence phenomena of localization may be suppressed.

### 3 Parallel Connexion of Crystals

We wish to construct a material element which combines the behavior of single crystals of various orientations and may thus be useful for the description of polycrystalline media. We do this by connecting the crystals in parallel. There are two motivations for this approach. First the parallel connexion is well established in the mechanics of materials. It is known that the most general linear viscoelastic behavior can be represented by a finite or infinite number of parallel elastic and Maxwell elements. Second the simplest homogenization procedure for a heterogeneous continuum consists of assuming a uniform straining throughout the representative volume element, thus giving the overall response as the mean value of the response of the individual components (procedure of Voigt in the elastic, Taylor in the plastic and Lin in the elastoplastic case), which is again equivalent to the formalism of a parallel connexion.

The single crystals are numbered by an index  $k$  ( $k = 1 \dots m$ ). The transplacement  $\mathbf{F}$  and the right Cauchy-Green tensor according to equation (2) are the same for the material element as well as for each of its constituents (crystals). The formula (3) remains valid for the material element and a similar formula holds for each constituent:

$$\mathbf{Z}_k = \mathbf{F}^{-1} \frac{\mathbf{T}_k}{\varrho} \mathbf{F}^{-T} \quad (35)$$

The total modified stress is obtained as the mean of the modified stresses of the constituents:

$$\frac{\mathbf{T}}{\varrho} = \frac{1}{m} \sum_{k=1}^m \frac{\mathbf{T}_k}{\varrho} \quad \text{or} \quad \mathbf{Z} = \frac{1}{m} \sum_{k=1}^m \mathbf{Z}_k \quad (36)$$

The multiplicative decomposition of the common transplacement  $\mathbf{F}$  into an elastic and a plastic part is different for each constituent:

$$\mathbf{F} = \mathbf{F}_{ek} \mathbf{F}_{pk} \quad (37)$$

We assume that all constituents are free of stress at time  $t = 0$ . The orientation of the crystal axes  $\mathbf{e}_{jk}$  of each of them with respect to the laboratory axes  $\mathbf{e}_j$  may at that time be described by an orthogonal tensor:

$$\mathbf{Q}_k = \sum_{j=1}^3 \mathbf{e}_{jk} \otimes \mathbf{e}_j \quad (38)$$

It is most useful to choose the placement at time  $t = 0$  as the reference placement of the material element (and hence of each of its constituents), i.e.  $\mathbf{F}(t = 0) = \mathbf{1}$ , but to choose as the reference placements of all lattices those in which the crystal axes and the laboratory axes coincide. At time  $t = 0$ , the lattice of the constituent  $k$  presents itself to the observer as

$$\mathbf{Q}_k = \mathbf{F}_{ek}(t = 0) = \mathbf{F}(t = 0) \mathbf{F}_{pk}^{-1}(t = 0) = \mathbf{F}_{pk}^{-1}(t = 0) \quad \implies \quad \mathbf{F}_{pk}(t = 0) = \mathbf{Q}_k^T \quad (39)$$

The advantages of this choice of an identical reference placement for all lattices — with reference crystal axes  $\mathbf{g}_j = \mathbf{e}_j$  — are:

- The elastic stiffnesses  $\mathcal{C}$  of all constituents according to equation (19) are the same.
- The four directions of the plastic sliding in the lattice reference placement are the same for all constituents, namely  $\mathbf{m}_j = (\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3)/\sqrt{3}$ .

The analysis of the behavior of the various constituents is therefore based on the same equations, differing only in the initial values of the internal variables  $\mathbf{F}_{pk}$  according to equation (39).

Let us introduce the abbreviation  $\mathbf{N}_k = \mathbf{F}_{pk}^{-1}$ . In the light of Noll's (1972) theory of simple materials the actual state of our material element is given by the external variable of state  $\mathbf{C}$  and the set of internal variables of state  $\mathbf{N}_1 \dots \mathbf{N}_m$ . The equation of state reads — see equations (36) and (5) with equation (9), where  $\mathbf{F}_p$ ,  $\mathbf{Z}_e$  and  $\mathbf{Z}$  have to be replaced by  $\mathbf{F}_{pk}$ ,  $\mathbf{Z}_{ek}$  and  $\mathbf{Z}_k$ , respectively —

$$\mathbf{Z} = \frac{1}{m} \sum_{k=1}^m \mathbf{N}_k \left( \mathcal{C} : \frac{1}{2} (\mathbf{N}_k^T \mathbf{C} \mathbf{N}_k - \mathbf{1}) \right) \mathbf{N}_k^T \quad (40)$$

and there are  $m$  tensorial evolution equations for the internal variables of the type

$$\mathbf{N}_k^{-1} \dot{\mathbf{N}}_k = - \sum_{j=1}^4 \kappa_{jk} (\mathbf{N}_k^T \mathbf{C} \mathbf{N}_k, \mathbf{N}_k^T \dot{\mathbf{C}} \mathbf{N}_k) \mathbf{m}_j \otimes \mathbf{n}_j (\mathbf{N}_k^T \mathbf{C} \mathbf{N}_k) \quad (41)$$

These equations are uncoupled so that the stress of each constituent can be derived separately once the function  $\mathbf{C}(t)$  is given.

One might think of further improving the material description by allowing an interaction of the constituents, i.e. attributing to each of them a separate value  $\mathbf{F}_k$  of the transplacement. This could be done on the basis of a spatial arrangement in a representative volume element or in the sense of a self-consistent approach (cf. Lipinski and Berveiller, 1989). However, since the problem is infinitesimally non-linear — the active mechanisms of the constituents are not a priori known — and all evolution equations become coupled, the resulting material element would be rather clumsy and the computational effort would increase enormously. So we rest content with the parallel connexion.

The question is most important, how many constituents have to be connected in parallel in order to give useful results. This shall be tested here by requiring isotropic elastic and plastic behavior from a virgin state. As long as the deformation is small and hence elastic, the tensors  $\mathbf{N}_k$  keep their initial values  $\mathbf{Q}_k$  according to equation (39) and the derivative of equation (40) reads

$$\dot{\mathbf{Z}} = \frac{1}{m} \sum_{k=1}^m \mathbf{Q}_k \left( \mathcal{C} : \mathbf{Q}_k^T \dot{\mathbf{E}} \mathbf{Q}_k \right) \mathbf{Q}_k^T \quad (42)$$

The right-hand side is an isotropic function of  $\dot{\mathbf{E}}$ , if according to equation (19) the following condition holds — see Böhlke and Bertram (1999) —

$$\frac{1}{m} \sum_{k=1}^m \sum_{i=1}^3 \mathbf{Q}_k \mathbf{g}_i \otimes \mathbf{Q}_k \mathbf{g}_i \otimes \mathbf{Q}_k \mathbf{g}_i \otimes \mathbf{Q}_k \mathbf{g}_i = \frac{2}{5} \mathcal{I} + \frac{1}{5} \mathbf{1} \otimes \mathbf{1} \quad (43)$$

We will compare two cases:

1) Case  $m = 12$ . A solution of equation (43) has been given by Gaffke and Heiligers — cf. Böhlke and Bertram (1998). The twelve orthogonal tensors  $\mathbf{Q}_k = 2\mathbf{n}_k \otimes \mathbf{n}_k - \mathbf{1}$  are symmetric and hence represent rotations of  $180^\circ$ . The components of the twelve unit vectors  $\mathbf{n}_k$  are chosen as  $(a \pm b \pm b)$ ,  $(b \pm a \pm b)$ ,  $(b \pm b \pm a)$ . Due to reasons of symmetry at least cubic elastic behavior is guaranteed, and isotropic behavior results for the special choice  $a = \sqrt{\frac{1}{2} - \frac{1}{30} \sqrt{25 + 10\sqrt{130}}} \approx 0.32708$ ,  $b = \sqrt{(1 - a^2)/2}$ .

2) Case  $m = 8$ . A solution of equation (43) is obtained if the 8 orthogonal tensors  $\mathbf{Q}_k$  represent rotations around the 8 space diagonal directions. Isotropic behavior results if the angle of rotation is  $36.83^\circ$ .



Figure 2. Eight (- - -) and twelve (—) constituents. Deviation from plastic isotropy in strain controlled tension tests under various angles  $\alpha$ .



Figure 3. Comparison of the 12 constituent model (—) with the Prandtl-Reuss model (- - -). Elastic-plastic transition under stretching  $\mathbf{D} = \text{diag}(2, -1, -1)$  and sudden change to  $\mathbf{D} = \text{diag}(-1, 2, -1)$ . The principal axes are the crystal axes.

The initial elastic behavior of the two material elements with 8 or 12 constituents is exactly isotropic but the plastic behavior is not, of course. Figure 2 shows the results of a deformation process which in the isotropic case should produce a uniaxial tension  $\sigma_{11}$ . Actually the value of this stress depends on the inclination of the tension with respect to the laboratory axes and stresses  $\sigma_{22} = -\sigma_{33} \neq 0$  and  $\sigma_{12} \neq 0$  are also present. The relative deviation is below 2.5% in the case of 12 constituents and about 7.5% in the case of only 8 constituents. So less than 12 constituents seem to yield insufficient results.

Figure 3 shows a comparison of the 12 constituent model with a Prandtl-Reuss model according to equations (33), (34). In the regions of steady state plastic flow the results coincide, but we notice marked deviations in the description of elastic-plastic transition phenomena.



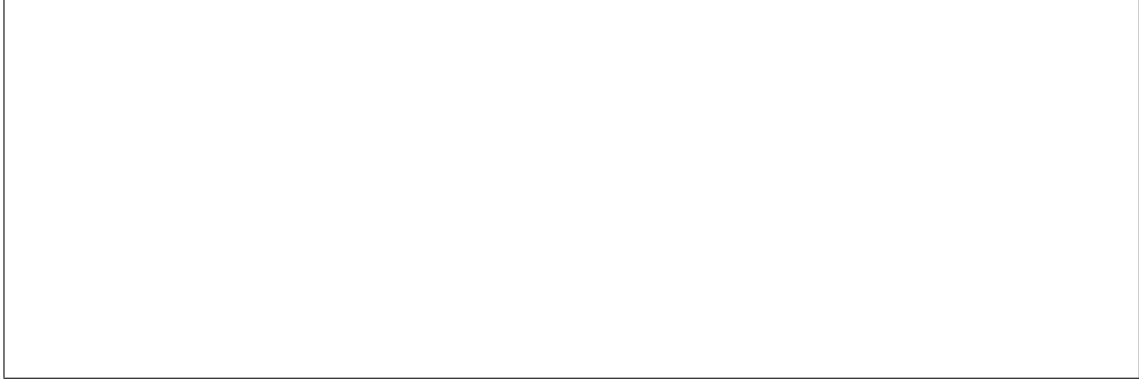


Figure 4. Model with 12 constituents: Study of the influence of the rolling texture on the elastic anisotropy by strain controlled tension tests. Young's modulus (- - - : Isotropic virgin state, — : State after a thickness reduction to 20%)

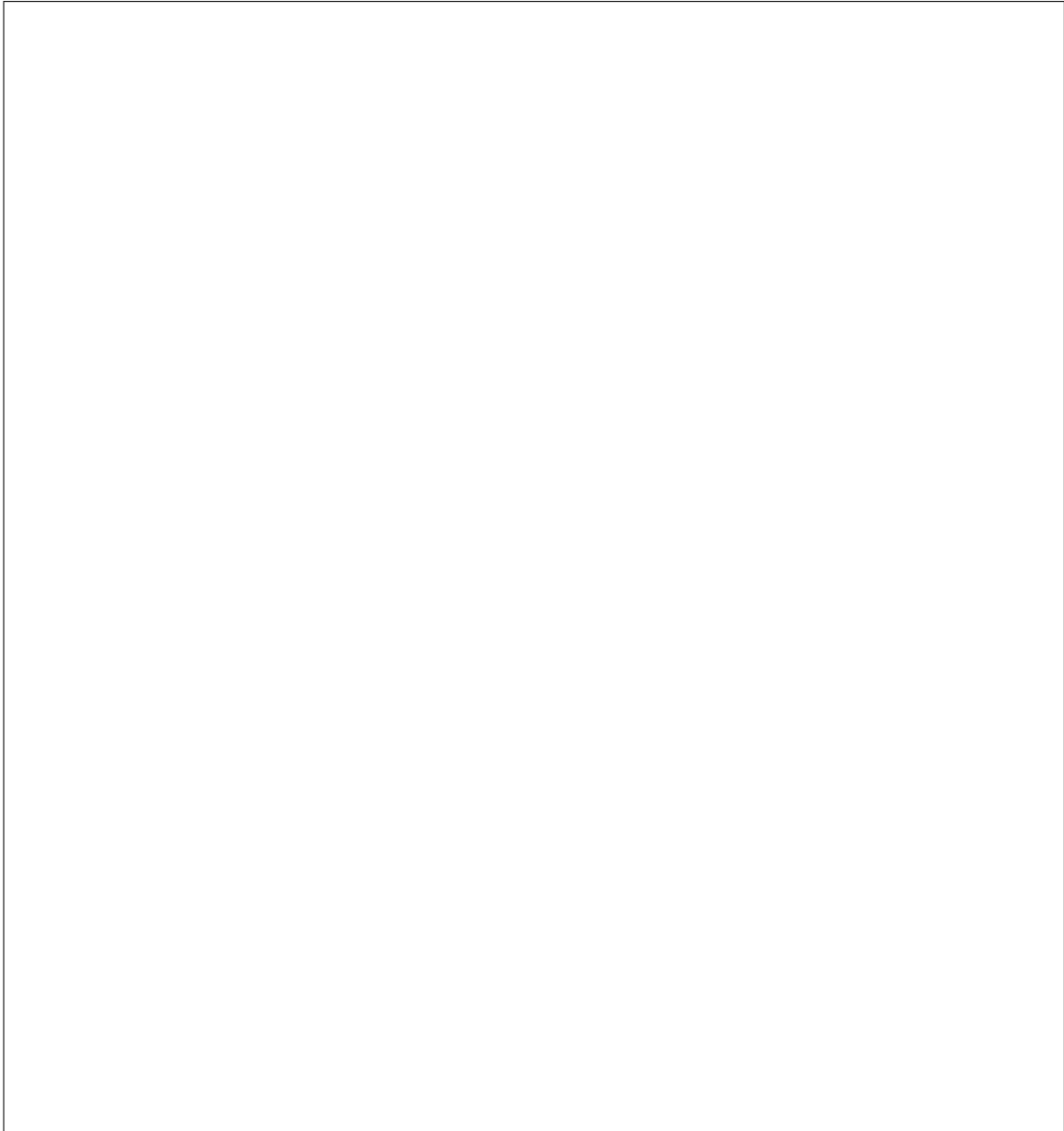


Figure 5. Model with 12 constituents: Study of the influence of the rolling texture on the plastic anisotropy by strain controlled tension tests. Stresses after 2.5% of plastic straining (- - - : "Isotropic" virgin state, — : State after a thickness reduction to 20%)

Finally, figures 4 and 5 show that the 12 constituent model predicts the formation of elastic and plastic anisotropy during a process of rolling. By the way, the evolution equations of only 3 of the 12 constituents have to be solved during the simulation of the rolling process because of its symmetry with respect to the laboratory axes.

## 4 Outlook

We have studied a constitutive law which takes into account some features of a crystalline aggregate. The intention was not to model all aspects of crystal plasticity, but to create a constitutive description which is more realistic than the classical one in that it shows a marked elastic-plastic transition (which is essential for the prediction of springback), a natural development of anisotropy, and softer behavior due to vertex-yield in each constituent (which is essential for the prediction of failure). We have seen that the proposed law is able to cover the items 5, 6, and 7 of the list in the introductory section. (That item 8 is also covered will be shown elsewhere.)

The material element consists of a parallel connexion of crystal-like constituents. These constituents are characterized by rather simple yield conditions and sliding mechanisms together with a cubic elastic law. But no elastic range, no yield condition and no plastic strain have been defined for the element as a whole, and this is just the reason why it allows the modelling of complex elastic-plastic behavior and is thus superior to the classical laws. The evolution of overall elastic and plastic anisotropy is a result of the development of a texture caused by the plastic deformation of the constituents.

The number of internal variables (9 components of each of the tensors  $\mathbf{N}_1 \dots \mathbf{N}_m$ , i.e. 108 scalar quantities in the case  $m = 12$ ) is much larger than that of a classical elastic-plastic law and so is the time required for the integration of the evolution equations. On the other hand, the effort would be even greater, if interactions of the crystals were taken into account.

An implementation of the model into a computer code will reveal whether the computation times are acceptable and how many constituents are really needed for reliable predictions.

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