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The reaction stresses of an incompressible second-gradient fluid

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Assumption of **incompressibility**:

$$0 = \mathbf{v} \cdot \nabla = \mathbf{1} : \mathbf{v} \otimes \nabla = \mathbf{1} : \mathrm{sym}[\mathbf{v} \otimes \nabla]$$

Simple material (First-gradient material)

Power per unit volume:

$$\pi = \mathbf{T} : \mathrm{sym}[\mathbf{v} \otimes \nabla]$$

Incompressible fluid (π is dissipated):

$$\mathbf{T} = \mathbf{T}_D(\mathrm{sym}[\mathbf{v} \otimes \nabla]) + \mathbf{T}_R$$

$$\mathbf{T}_R = -p\,\mathbf{1}$$

The pressure p is a scalar **reaction field**

Linear isotropic behaviour (Newton, Navier-Stokes):

$$\mathbf{T}_D(\mathrm{sym}[\mathbf{v} \otimes \nabla]) = 2\eta\,\mathrm{sym}[\mathbf{v} \otimes \nabla]$$

Important observation of St-Vénant (1869):
The description of complicated flows may need the inclusion of higher velocity gradients
This implies that the material behaviour contains a characteristic length

Second-gradient fluid:

$$\pi = \mathbf{T} : \mathbf{v} \otimes \nabla + \mathsf{T} \,\vdots\, \mathbf{v} \otimes \nabla \otimes \nabla$$

Incompressibility additionally implies:

$$\mathbf{0} = \nabla(\mathbf{v} \cdot \nabla) = \mathbf{1} : \mathbf{v} \otimes \nabla \otimes \nabla$$

$$\Longrightarrow \qquad 0 = (\mathbf{q}\cdot\nabla)(\mathbf{v}\cdot\nabla) = \mathbf{1}\otimes\mathbf{q} \,\vdots\, \mathbf{v}\otimes\nabla\otimes\nabla = \mathrm{sym}^{[23]}[\mathbf{1}\otimes\mathbf{q}] \,\vdots\, \mathbf{v}\otimes\nabla\otimes\nabla$$

Constitutive equations:

$$\begin{aligned} \mathbf{T} &= \mathbf{T}_D(\mathbf{v} \otimes \nabla, \mathbf{v} \otimes \nabla \otimes \nabla) + \mathbf{T}_R \\ \mathsf{T} &= \mathsf{T}_D(\mathbf{v} \otimes \nabla, \mathbf{v} \otimes \nabla \otimes \nabla) + \mathsf{T}_R \end{aligned}$$

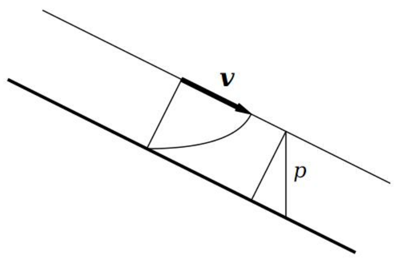
$$\Longrightarrow \qquad \mathbf{T}_R = -p\mathbf{1}\,, \qquad \mathsf{T}_R = \mathrm{sym}^{[23]}[\mathbf{1} \otimes \mathbf{q}]$$

Four scalar **reaction fields**: $p, \mathbf{q}$

A trivial example: Flow down an inclined plane

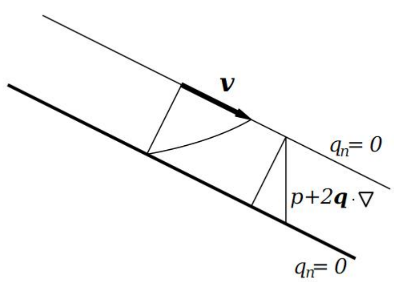
Linear isotropic case

Simple fluid:



The scalar reaction field p is unique

Second-gradient fluid:



The scalar reaction field $p + 2\,\mathbf{q} \cdot \nabla$ is unique

The vector field $\mathbf{q}$ is almost arbitrary. If $\mathbf{q}$ is chosen then p is known

The stress fields $\mathbf{T}$ and T are not unique

So the situation is unsatisfactory

Remedy: Recourse to the weakly compressible fluid

Split of the first velocity gradient into a deviatoric and a spherical part

$$\mathbf{v} \otimes \nabla = (\mathbf{v} \otimes \nabla)^* + \frac{1}{3} (\mathbf{v} \cdot \nabla)\, \mathbf{1} \qquad \text{with} \qquad (\mathbf{v} \otimes \nabla)^* : \mathbf{1} = 0$$

Equation of continuity

$$\varrho^\bullet \equiv \frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot (\nabla \varrho) = -\varrho\, (\mathbf{v} \cdot \nabla) \qquad \Longrightarrow \qquad \mathbf{v} \cdot \nabla = -\frac{\varrho^\bullet}{\varrho}$$

$$\nabla(\mathbf{v} \cdot \nabla) = -\frac{1}{\varrho}(\nabla \varrho)^\bullet - \frac{1}{\varrho}(\nabla \otimes \mathbf{v})^* \cdot \nabla \varrho + \frac{4\varrho^\bullet}{3\varrho^2} \nabla \varrho$$

Viscous stresses

Representation theorems of the general hemitropic constitutive laws

$$\mathbf{T}_v = \mathrm{sym}[\alpha_1\,(\mathbf{v}\cdot\nabla)\,\mathbf{1} + \alpha_2\,\mathbf{v}\otimes\nabla\underline{+\alpha_8\,\boldsymbol{\epsilon} : \mathbf{v}\otimes\nabla\otimes\nabla}]$$

$$\begin{aligned}\mathsf{T}_v \;=\; & \mathrm{sym}^{[23]}\Big[\alpha_3\,\mathbf{1}\otimes\nabla(\mathbf{v}\cdot\nabla) + \frac{\alpha_4}{2}(\nabla(\mathbf{v}\cdot\nabla)\otimes\mathbf{1} + \mathbf{1}\otimes\Delta\mathbf{v}) + \alpha_5\,\Delta\mathbf{v}\otimes\mathbf{1} \\ & +\alpha_6\,\mathbf{v}\otimes\nabla\otimes\nabla + \alpha_7\,\nabla\otimes\mathbf{v}\otimes\nabla\underline{+\alpha_9\,\boldsymbol{\epsilon}\cdot\mathrm{sym}[\mathbf{v}\otimes\nabla]}\Big]\end{aligned}$$

We disregard the underlined coupling terms and obtain isotropic behaviour

$$\mathbf{T}_v = \alpha_2\,\mathrm{sym}[(\mathbf{v}\otimes\nabla)^*] - \Big(\alpha_1 + \frac{\alpha_2}{3}\Big)\,\frac{\varrho^\bullet}{\varrho}\,\mathbf{1}$$

$$\begin{aligned}\mathsf{T}_v = \;& \mathrm{sym}^{[23]}\Big[\frac{\alpha_4}{2}\mathbf{1}\otimes(\mathbf{v}\otimes\nabla)^*\cdot\nabla + \alpha_5\,(\mathbf{v}\otimes\nabla)^*\cdot\nabla\otimes\mathbf{1} \\ & +\alpha_6\,(\mathbf{v}\otimes\nabla)^*\otimes\nabla + \alpha_7\,(\nabla\otimes\mathbf{v})^*\otimes\nabla \\ & +\Big(\alpha_3 + \frac{\alpha_4}{6} + \frac{\alpha_7}{3} + \frac{\alpha_6}{3}\Big)\,\mathbf{1}\otimes\Big(-\frac{1}{\varrho}(\nabla\varrho)^\bullet - \frac{1}{\varrho}(\nabla\otimes\mathbf{v})^*\cdot\nabla\varrho + \frac{4\varrho^\bullet}{3\varrho^2}\nabla\varrho\Big) \\ & +\Big(\frac{\alpha_4}{2} + \frac{\alpha_5}{3}\Big)\Big(-\frac{1}{\varrho}(\nabla\varrho)^\bullet - \frac{1}{\varrho}(\nabla\otimes\mathbf{v})^*\cdot\nabla\varrho + \frac{4\varrho^\bullet}{3\varrho^2}\nabla\varrho\Big)\otimes\mathbf{1}\Big]\end{aligned}$$

A material that is completely characterized by these viscous constitutive laws would have a strange behaviour: Given a constant state of stress, it could change its volume without limit

Elastic stresses

Therefore we assume, that a deviation of the mass density ϱ from some value ϱ_0 is accompanied by a storage of elastic energy

$$w = w(\varrho, |\nabla\varrho|)\,, \qquad w^\bullet = \frac{\partial w}{\partial \varrho}\varrho^\bullet + \frac{\partial w}{\partial |\nabla\varrho|}|\nabla\varrho|^\bullet$$

with

$$\varrho^\bullet = -\varrho\,\mathbf{1} : \mathbf{v}\otimes\nabla$$

$$|\nabla\varrho|^\bullet = -\,|\nabla\varrho|\big(\mathbf{1} + \mathbf{e}\otimes\mathbf{e}\big) : \mathbf{v}\otimes\nabla - \varrho\,\mathbf{e}\cdot\nabla(\mathbf{v}\cdot\nabla)\,, \qquad \mathbf{e} \equiv \frac{\nabla\varrho}{|\nabla\varrho|}$$

The power of the elastic stresses per unit volume is then

$$\pi_e = \mathbf{T}_e : \mathbf{v}\otimes\nabla + \mathsf{T}_e \,\vdots\, \mathbf{v}\otimes\nabla\otimes\nabla = \varrho w^\bullet$$

We infer

$$\mathbf{T}_e = -\frac{\partial w}{\partial\varrho}\,\varrho^2\,\mathbf{1} - \underline{\frac{\partial w}{\partial|\nabla\varrho|}\,\varrho\,|\nabla\varrho|}\big(\mathbf{1} + \mathbf{e}\otimes\mathbf{e}\big)$$

$$\mathsf{T}_e = -\frac{\partial w}{\partial|\nabla\varrho|}\,\varrho^2\,\mathrm{sym}^{[23]}[\mathbf{1}\otimes\mathbf{e}]$$

Let $|\varrho - \varrho_0|$ and $|\nabla\varrho|$ be small, so that the following quadratic form is appropriate

$$w = \frac{K}{2\varrho_0^3}\Big((\varrho-\varrho_0)^2 + L^2\,|\nabla\varrho|^2\Big)$$

K: modulus of compressibility, $\qquad$ L: characteristic elastic length

The stresses become linear in $\varrho - \varrho_0$ and $|\nabla\varrho|$ if the underlined higher order term is disregarded and ϱ is approximated by ϱ_0

$$\mathbf{T}_e = -\frac{K}{\varrho_0}(\varrho-\varrho_0)\mathbf{1}$$

$$\mathsf{T}_e = -\frac{KL^2}{\varrho_0}\mathrm{sym}^{[23]}[\mathbf{1}\otimes\nabla\varrho]$$

Total stresses

We choose a parallel connection of the elastic and viscous stresses in the sense of Kelvin-Voigt

$$\mathbf{T} = \mathbf{T}_v + \mathbf{T}_e\,, \qquad \mathsf{T} = \mathsf{T}_v + \mathsf{T}_e$$

Moreover, we define the scalar field

$$u \equiv \frac{K}{\varrho_0}\,(\varrho - \varrho_0)$$

and the two retardation times corresponding to the second and third order constitutive equation

$$t_2 \equiv \frac{\alpha_1}{K}\,, \qquad t_3 \equiv \frac{\alpha_3}{KL^2}$$

The denominators ϱ in the viscous stresses are replaced by ϱ_0. So we arrive at

$$\mathbf{T} = \alpha_2\,\mathrm{sym}\Big[(\mathbf{v}\otimes\nabla)^*\Big] - \Big(\Big(\frac{\alpha_2}{3K} + t_2\Big)u^\bullet + u\Big)\mathbf{1}$$

$$\begin{aligned}\mathsf{T} \;=\; & \mathrm{sym}^{[23]}\Bigg[\frac{\alpha_4}{2}\mathbf{1}\otimes(\mathbf{v}\otimes\nabla)^*\cdot\nabla + \alpha_5\,(\mathbf{v}\otimes\nabla)^*\cdot\nabla\otimes\mathbf{1} + \alpha_6\,(\mathbf{v}\otimes\nabla)^*\otimes\nabla + \alpha_7\,(\nabla\otimes\mathbf{v})^*\otimes\nabla \\ & + \mathbf{1}\otimes L^2\Bigg(\Big(t_3 + \frac{1}{KL^2}\Big(\frac{\alpha_4}{6} + \frac{\alpha_7}{3} + \frac{\alpha_6}{3}\Big)\Big)\Big(-(\nabla u)^\bullet - (\nabla\otimes\mathbf{v})^*\cdot\nabla u + \frac{4}{3K}u^\bullet\nabla u\Big) - \nabla u\Bigg) \\ & + \frac{1}{K}\Big(\frac{\alpha_4}{2} + \frac{\alpha_5}{3}\Big)\Big(-(\nabla u)^\bullet - (\nabla\otimes\mathbf{v})^*\cdot\nabla u + \frac{4}{3K}u^\bullet\nabla u\Big)\otimes\mathbf{1}\Bigg]\end{aligned}$$

Limiting process to the incompressible fluid

There will hardly be any real material that is totally incompressible. However, the modulus of compressibility K may be very high so that only small changes of mass density will occur.
We let K tend towards infinity. The stresses and hence the values of u can only remain finite if $K \to \infty$ implies $\varrho \to \varrho_0$. We obtain

$$\mathbf{T} = \mathrm{sym}[\alpha_2\,(\mathbf{v} \otimes \nabla)] - \,p\mathbf{1}$$

$$\begin{aligned} \mathsf{T} \;&=\; \mathrm{sym}^{[23]}\Big[\frac{\alpha_4}{2}\mathbf{1} \otimes \Delta\mathbf{v} + \alpha_5\,\Delta\mathbf{v} \otimes \mathbf{1} + \alpha_6\,\mathbf{v} \otimes \nabla \otimes \nabla \\ &\qquad + \alpha_7\,\nabla \otimes \mathbf{v} \otimes \nabla + \mathbf{1} \otimes 2\mathbf{q}\Big] \end{aligned}$$

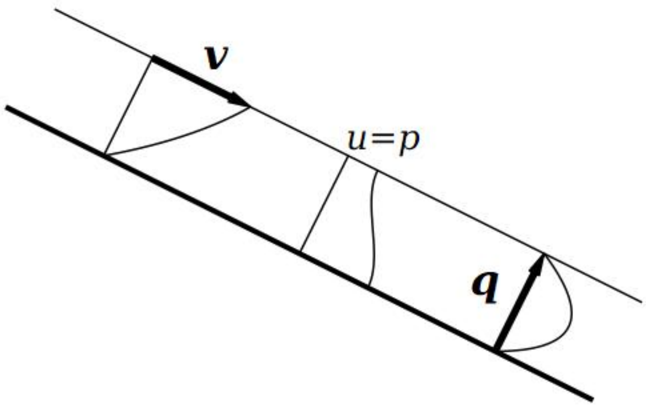
with

$$p = t_2\,u^\bullet + u$$

$$2\mathbf{q} = -\,L^2\,\Big(t_3\Big((\nabla u)^\bullet + (\nabla \otimes \mathbf{v}) \cdot \nabla u\Big) + \nabla u\Big)$$

Now, the fields p and $\mathbf{q}$ do not represent four unknown scalar functions but are derived from only one scalar function u. This makes it possible to obtain information on the internal stresses, the surface tension and the forces acting on fixed boundaries that is not available with the classical approach.

Result if L is one third of the fluid thickness



Note: We study strictly isochoric motions. No viscoelastic volume change is allowed. Nevertheless, the limit from the weakly compressible case allows the reduction to one function u. Its contribution to the stresses makes use of three viscoelastic constants L, t_2, t_3 which describe the behaviour under volumetric changes, although these changes were eliminated by our limiting process.

Thank you for your attention!

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